What is Mathematics? – a Short Introduction to the Contemporary Thinking about Mathematics

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Abstract

To some extent we all (have to) deal with Mathematics. It provides indispensable tools for any kind of scientific research, engineering work and it also permeates our everyday life. But the question, 'What is Mathematics anyway?', remained unanswered since the very beginning of Mathematics (and Philosophy). This does not mean that we do not have any answer. In fact, on the contrary, we have many answers. Here we briefly survey the traditional answers (platonism vs. empirism, logicism, formalism, intuitionism, structuralism) and then review the recent approaches based on the physically embodied and evolutionary view of human thinking (e.g. math as gossiping, math and metaphors).

'What is Mathematics? – this is an interesting philosophical problem, probably as old as Mathematics itself. Clearly, it is not a mathematical question: the possible answer will not appear as the result of some calculation or by constructing a mathematical object. Moreover, it is definitely not a practical question. It rather seems to be an unimportant problem, as we do indeed know how to pursue Mathematics. There are well-written textbooks containing knowledge that can now be considered eternal and from them we can learn the techniques and there are numerous open problems where someone can start research. Also, there is an established international reviewing system for examining the attempts of solving those problems and judging the importance of new results. Therefore the question may sound philosophical in the vulgar sense as well: trying to answer is just a pastime activity, amusing at most. On the contrary, we claim that thinking about the conceptual roots of Mathematics is fundamental for understanding and teaching it. Many misunderstandings, fear, statements like “I don't like math.”, “I'm not good at math.”, etc. come from the fact that most people have wrong ideas about the nature of Mathematics, usually some leftover of a hidden philosophical perspective in their thinking.

First we quickly review the traditional ways of thinking about Mathematics.Then we proceed with outlining more recent ideas in a form of short book reviews.

1 Traditional Approaches

If we would like to simplify the traditional approaches of the philosophy of Mathematics we can separate two main opposite views, usually termed as the *platonist* and the *empirical* standpoints.

Platonism The most prevalent way of thinking about Mathematics. It is sort of agreed that in order to be a working mathematician one has to believe in the independent existence of mathematical objects, very much like how Plato thought about abstract ideas, hence the term, *platonism.* For this view the basic problem is that it is difficult to explain how we can have knowledge about things in a different, nonmaterial world as directly we can only perceive the world around us. Also, a platonist still lacks an explanation why Mathematics has so many applications in real-world and how exactly things partake in the nature of mathematical ideas.

Empirism The opposite of platonism. Empirism says that all (mathematical) knowledge comes from observation, experience through empirical induction. For instance we could measure the inside angles of many triangles and we observe that their sum is $180°$, so we conclude that we have a mathematical law. However, this is not the way we do Mathematics. In math we can prove things. We can have absolute certainty. Taking the above statement as an empirical law, we can never be sure that the next triangle will not have a different sum. As a proven mathematical theorem, the truth of the statement follows necessarily and it is universal.

In addition to the two major players described above there are other directions for thinking on Math.

Logicism According to logicism, Mathematics is an extension to Logic. While it is certainly true that logic is foundational for Mathematics, it is already a big venture to describe elementary arithmetics as formal logical statements [11]. For example, describing algebraic topology simply with logical formulas would be like describing an animal in terms of the elementary particles instead of organs and its ecological environment.

Formalism Mathematics is without any meaning, just manipulation of symbols, i.e. proving theorems from axioms by logical inference. The basic axioms are chosen so that other sciences can interpret them. The Hilbert Programme: Carefully and rigorously formalize each branch of Mathematics, together with its logic, and then to study the formal systems to make sure they are coherent. Emphasis is on finite methods for proofs. The fall: Gödels Incompleteness Theorem. Formalism is still a common retreat: What is a complex number? I dont know, but I can calculate with pairs of real values according to the given operations.

Intuitionism Mathematics is primarily a mental activity, so it is a construct of human brains. Metaphysical assumptions should be removed from Math, therefore the Law of Excluded Middle should not be used, since it presupposes the existence of truth values for every statement, thus indirect proofs cannot be used. Language is only a medium to communicate mathematical knowledge. Followers of intuitionism ended up with a different, more restrictive logic than the classical one, and thus a different incompatible constructive Mathematics.

Structuralism Mathematics is the science of patterns (structures) [8]. The objects in the positions of the patterns do not matter. Transition from objects to relations: the essence of a natural number is its relations to other natural numbers.

A detailed description of these ideas can be found in [10].

2 Recent Advances

Mathematics

The traditional approaches of philosophy of Mathematics are partially outdated, since they tend to ignore and not to discuss recent developments of Mathematics. Mathematics has indeed developed a lot, especially recently. Just to name a few of these changes:

Foundation As mathematical foundations the usual choice is either set theory or logic. Both are special cases of new branch of Mathematics, called *category theory*, outgrown from algebraic topology. It also emphasizes the transition from static structures to processes (structure-preserving mappings). For a sweeping view of whole mathematics from a categorical viewpoint see [7].

Computation The rise of the amount of available computational power is indeed changing how Mathematics is studied and researched. The computer is like the microscope in biology or the telescope for astronomy, we can see things that we could not glimpse before. It is not different from using pen and paper (some kind of external representation that allows easier manipulation of mathematical objects), but its sheer power takes computation to a different level. Also, the theory of computation revealed that there are undecidable problems, i.e. questions that provably have no answers.

Proof The notion of an exact proof has become a bit blurred in at least two different ways:

- The proof consists of a *systematic check of many cases performed by a computer*. Due to the number of cases involved in the proof, it is not comprehensible just by looking at it, though one can fully understand the algorithm producing the proof. For example: the four-color problem in graph theory [13].

- The proof is *so long and complicated that no single person can understand and verify it in its totality.* The prime example is the Classification Theorem of Finite Simple Groups. The proof has a long history [3, 9] spreading over several hundred journal articles. Recently appeared textbooks try to summarize and simplify the proof (e.g. [12]), to make it digestible for new generations of mathematicians. Without this effort humanity would loose mathematical knowledge, even in the 21st century.

Chance and irregularity Fractal geometry, chaos theory – the mathematics of the irregular shapes and processes brings new topics that were previously thought not amenable to mathematical treatment. Probability, the math of chance is also a relatively new field.

Cognitive Science

Cognitive science is the interdisciplinary research of the mind and intelligence. Clearly the mind is very complex multifaceted phenomenon so its study requires several disciplines like computer science, philosophy, psychology, artificial intelligence, neuroscience, linguistics, and anthropology. Though it is far from answering all questions cognitive science provides lots of new insights how the mind and the brain does thinking in general and Mathematics in particular.

embodiment of mind Our way of thinking, the human concepts are structured and constrained by the way we operate in the 3-dimensional physical world using our body in everday life. This sounds obvious but classical AI completely ignored this embodiment of the mind and focused only on very high-level mental functions like playing board games.

cognitive unconscious Most of the computational work is done by the brain below the conscious level, and we can not look directly at these low-level thought processes.

metaphorical thought Metaphors are not just poetic tools, but they seem to be the very basic mechanisms of human cognition, i.e. understanding something in terms of another thing.

3 Metaphors

George Lakoff and Rafael E. Núnez: **Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being.**, 2000. [6]

It is common to distinguish between humanities and the so called hard sciences and divide people into two categories according to that separation. From this viewpoint the poet and the scientist take different sides, they think differently. It seems that this division is fundamentally wrong. In their seminal work [5] cognitive linguists George Lakoff and Mark Johnson claim that "…human

Source Domain OBJECT COLLECTION		*Target Domain* ARITHMETIC
Collections of objects of the same size	$\rightarrow$	Numbers
The size of the collection	$\rightarrow$	The size of the number
Bigger	$\rightarrow$	Greater
Smaller	$\rightarrow$	Less
The smallest collection	$\rightarrow$	The unit(One)
Putting collections together	$\rightarrow$	Addition
Taking a smaller collection from a larger collection	$\rightarrow$	Subtraction

Figure 1: One of the grounding metaphors for arithmetic. Integer numbers are understood as collection of objects, like putting things into piles.

thought processes are largely metaphorical." Metaphor is not just a figure of speech, but "the essence of metaphor is understanding and experiencing one kind of thing in terms of another... reasoning about one kind of thing as if it were another." Metaphors construct our understanding of the world and they seamlessly guide our thinking. For instance the two different metaphors, 'Argument is War' versus 'Argument is Dance', convey two completely different interpretations of the activity of arguing. In the first one you aim to destroy someone else's opinion, while in the other one your are willing to learn or teach something new, the discussion is cooperative.

The basic idea of applying the cognitive metaphors to math is that *we understand elementary abstract concepts in terms of our sensory-motor experience and then we understand more complicated abstract objects in terms of simple abstract objects, and so on layers upon layers.* The fundamental cognitive abilities to ground Mathematics into sensory-motor experience are twofold:

nonmathematical cognitive mechanisms: basic spatial relations, groupings, motion, distribution of things in space, changes, bodily orientations, basic manipulations of objects (e.g. rotation, stretching), iterated actions, ...

number sense: We have very a very basic numerical ability, innate arithmetic: addition, substraction up to 3. Even babies and some animals have this ability. In the brain it is located at the junction of different modalities [1], e.g. recognizing that two beeps and two flashes of light have something common.

The extension of innate arithmetic is still directly grounded by the following 4 metaphors:

1. 'Arithmetic As Object Collection' (Fig. 1). We have an innate understanding of dealing with collections of objects. The extension of arithmetic follows easily from repeated actions: multiplication is repeated addition, division is repeated subtraction.

2. 'Arithmetic As Object Construction' Numbers understood as composite objects, constructed from other numbers.

3. The Measuring Stick Metaphor. Measuring length by counting its segments.

4. 'Aritmetic As Motion Along a Path' The origin of the number line.

Arithmetic has properties coming from all four domains, it is a blend of all these metaphors. Once we get acquainted with numbers, they can become the domain of other metaphors. For example in a sense functions behave like numbers as they can be added together. This is true in general for building mathematical theories. We start with basic notions derived from our experience directly, and after some time we get used to these abstractions, so their abstract nature disappears, they become familiar objects, thus we again have direct experience. Therefore we again build a metaphor to understand something new and more abstract using the now well understood previous target domain as source domain.

We can also give grounding metaphors for basic mathematical objects other than numbers. Logic is grounded in our everyday experience, in spatial reasoning: physical containment entails logical inference. Therefore Venn-diagrams are not just mathematical tools, but they show the origins of our mathematical thinking. The explanation of infinity is quite challenging, since one can argue that we have experience only of finite things. But it is not difficult to see that everyday continuous actions require iterated actions, e.g. walking require taking steps. Then infinity is conceptualized as a continuous process without an end.

The trouble is that usually these metaphors are not revealed in education, only the end result is introduced. Without their grounding ideas some theorems may be difficult to understand. For instance, students are presented with the equation

$$e^{\pi i} + 1 = 0$$

that cannot be understood as a statement claiming that two quantities are the same. Actually, some of the symbols are difficult to comprehend as mere quantities. The equation expresses something highly metaphorical. It is a network of mathematical ideas: cartesian plane, complex numbers, unit circle, polar coordinates, functions as numbers, trigonometry, recurrence as circularity, power series, etc.

4 Evolutionary Story

Keith Devlin (Stanford University), **The Math Gene – How Mathematical Thinking Evolved and Why Numbers are like Gossip**, 2000. [2]

Mathematics is a human activity, so when asking for the true nature of Mathematics it is sensible to investigate the origin of those creatures that are doing it. The current scientific explanation for the origin of our species is evolutionary.

Math as Gossiping

- for doing math we use some faculty of the brain that was evolved for something else (exaptation)
- language is off-line thinking
- gossiping is genuinely human and provides a mechanism for creating and maintining group commitment
- math is gossiping about abstract objects
- difficulty of dealing with abstractions

Figure 2: Key points of the argument.

Plants and animals evolved through billions of years on Earth and Biosciences are busy with detecting all the details of this slow process. We face greater difficulties when we try to explain the origin of human brain, human thinking, especially language. However, according to Keith Devlin, once we have the language, sooner or later mathematical abilities come for free, as Mathematics is nothing else but a very special language, or a specialized use of our language.

We have been doing Mathematics only for few thousands of years. Clearly, evolutionary development of humans is not possible during such a short time. Thus we can conclude that we use some ability of the brain which was an adaptation for some other survival task. When accepting this argument we still have to explain why it took so long for Mathematics to appear. Math needs a relatively developed society (with some form of economy and sciences, engineering knowledge, etc.) to leverage its advantages.

So what is the special capability of the brain that was used well before people started doing Math? The answer is simple: language and gossiping. Language is not surprising, Math is sometimes thought to be a special way of using the language, but gossiping needs some more explanation. It is a thoroughly checked fact that most human conversations are about gossiping. Even at a scientific conference, people talk mostly about other people. Talking about the complex tangled web of social relations, understanding the intricate meaning of events, detecting patterns of behaviour require a great deal of computational power. The evolutionary benefit of gossiping is immediate: the more you know the more you care, so gossiping reinforces the bond between group members.

The main claim of this book is that Mathematics is just gossiping about abstract objects:

> "To put it simply, mathematicians think about mathematical objects

> and the mathematical relationships between them using the same mental faculties that the majority of people use to think about other people.

If doing Mathematics requires only our language skills used in a special way, then how can we explain the fact that people do seem to have trouble with math. The difficulty lies in dealing with abstractions. We can distinguish 4 levels of abstraction.

- Level 1: no abstraction, the objects thought are real and perceptually accessible, but might well involve imagining different arrangements (online thinking, 'if this, then that').
- Level 2: objects are real, familiar, but not perceptually accessible in the immediate environment. (offline thinking)
- Level 3: objects never were never actually encountered, but they are combinations of properties of real objects. This is equivalent to having a language.
- Level 4: objects have no simple or direct connections to real objects. (mathematical thinking)

The first 3 levels pose no problems for normal people, but there seems to be a gap before reaching the fourth level. According to psychological experiments (Wason selection test), the same task is carried out with higher success rate by the subjects if it is in a well-known social context as opposed to some abstract settings. This hints that primary goal of math education should be the enhancement of abstraction skills.

5 Socio-Cultural Perspective

Reuben Hersh (University of New Mexico), **What Is Mathematics, Really?**, 1999. [4]

Another take on the issue that Mathematics is a human activity, but from a different angle, more on the cultural side. What is Mathematics? It is neither physical nor mental, it is social. It is part of culture, it is part of history. The author calls this perspective as 'humanism':

> ... *from the viewpoint of philosophy* Mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context. I call this viewpoint *humanist*.

The term is slightly unfortunate as it is loaded with other meanings, so one should think it as 'humanist philosophy of Mathematics'.

The existence of mathematical ideas

Whenever there is a new scientific theory, or in this case, a new philosophical perspective we have to test it on old problems. For instance the platonism versus anti-platonism debate. This is a prime example of a philosophical heritage that lies as dead weight on our western-european thinking. We can basically imagine two types of existence: mental and physical. Then the question arises whether Mathematics belongs to the former or the latter one. According to humanism we get the question completely wrong, there are other ways of existence. Concepts, once devised can exist independently from our minds, e.g. society, war, peace, etc. These are social concepts, and they all have mental and physical aspects, and we are usually not bothered with the question how do they exist? Similarly for Mathematics:

> "Mathematics consists of concepts. Not pencil or chalk marks, not physical triangles or physical sets, but concepts, which may be suggested or represented by physical objects."

> "Fact 1: Mathematical objects are created by humans. Not arbitrarily, but from activity with existing mathematical objects, and from the needs of science and daily life."

> "Fact 2: Once created, mathematical objects can have properties that are difficult for us to discover."

Fact 1 is in accordance with the metaphor based approach, as the basic mathematical notions come from or constrained by how we perceive the world around us.

> "The observable reality of Mathematics is this: an evolving network of shared ideas with objective properties."

Discovery or Creation

Is Mathematics created or discovered? Again, the humanist answer shows that the question is not really a sensible one.

> "When several mathematicians solve a well-stated problem, their answers are identical. They all discover that answer. But when they create theories to fulfill some need, their theories aren't identical. They create different theories."

Discovery or creation? Well, both. Or it depends. We are often discontent with these kind of answers. We expect a clear, one-sided, decisive answer. The Law of Excluded Middle is deeply engrained in our western thinking. However, reality is lot richer, it posseses several aspects.

6 Conclusion

Clearly, we are not able to provide a definite answer for the question 'What is Mathematics?', but maybe this short exposition of the attempts to answer it provides some insights for the reader. We surely tried to loosen up the very tight, rigid and strict understanding of Mathematics that we usually get in mainstream education. The reader is invited to think about the ideas presented here and not to believe them unconditionally. After some time the effect of these considerations can be evaluated. Do they help in understanding mathematical concepts? As a final advice, we finish with an allusion to a science-fiction blockbuster:

"Unfortunately, no one can be told what Mathematics really is. You have to see it for yourself."

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